EXAMPLES OF THE CONSTRUCTION OF RIEMANN'S SURFACES FOR THE INVERSE OF RATIONAL FUNCTIONS, BY THE METHOD OF CONFORMAL REPRESENTATION.

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With an Introduction by

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The subject of Riemann's surfaces is ordinarily introduced by examples of functions which come under the general type: $(z - a_1)^{\alpha_1} (z - a_2)^{\alpha_2} \ldots (z - a_n)^{\alpha_n}$ where $\alpha_1, \alpha_2, \ldots \alpha_n$ are rational numbers (cf. for instance Neumann's *Abelsche Integrale*). The surface in these cases can be constructed without difficulty, because it is easy to see how the different branches of the function go over into one another when we let z move around one or more of the points $a_1, a_2, \ldots a_n$. The method here used can theoretically be applied to the construction of Riemann's surfaces for any algebraic function explicit or implicit. In all except the simplest cases, however, the work of following the different branches of the function from one point to another becomes so complicated that the method, though essential to the proof of the *existence* of a Riemann's surface, is of little practical use in its *construction*, at least until it has been perfected by the addition of some method (for instance that of Puiseux) for the discussion of the singular points.

When, however, the algebraic function for which the Riemann's surface is to be constructed is the inverse of a single valued i. e. of a rational function a different and very instructive method of construction can be used. This method consists in mapping the images of the various sheets of the Riemann's surface on the plane of the independent variable and from the figure thus obtained deducing the connection of the sheets. I believe that this method is eminently suited to elementary instruction. I know, however, of no place where it is clearly set forth except in Klein's lectures on the Theory of Functions, delivered in Leipzig in 1880–81 (lithographed 1892),* and even here the number of examples treated is too small for the needs of the American student. I hope, therefore, that the following treatment of the subject by Mr. Bouton, accompanied as it is by a rather extensive set of illustrative examples, some of

*In almost all books on the theory of functions the conformal representation of the sheets of Riemann's surfaces is considered more or less explicitly. This method of conformal representation is not, however, usually regarded as a means of constructing the Riemann's surface, nor is it treated in as elementary and extensive a manner as it would seem to deserve. As to the importance of this method for *transcendental* functions cf. for instance Klein-Fricke: *Modulfunktionen*.

which are extremely elegant, will prove a welcome addition to the literature of the subject.*

The following is an abstract of the principles used:

Let $w = f(z)$ be the rational function whose inverse is to be studied. We consider two planes in which we represent the complex quantities z and w and which correspond to one another point for point by means of the above relation. Instead of these planes it is frequently more convenient to consider two spheres obtained from the planes by stereographic projection. The advantage gained by this is that $z = \infty$ and $w = \infty$ correspond to definite points on the spheres (the north poles). Although this is not the case for the planes, it is customary even here to speak of the *point at infinity* in each plane as corresponding to these values. It is well known that angles are not changed by stereographic projection, so that the angle between any two intersecting curves on either of the spheres and the angle between their stereographic projection on the corresponding plane will be the same. An exception of course occurs when the curves intersect at the north pole of the sphere, for in this case the stereographic projections do not intersect at all. Even here, however, in order to avoid exceptions it is customary to say that the two curves in the plane make *at infinity* the same angle which their stereographic projections made at the north pole of the sphere. By means of these conventions it is possible to avoid the use of spheres for most purposes.

The theorem upon which all that follows is based is that an angle in the z-plane is equal to the corresponding angle in the w-plane except when the vertex of the angle in the z-plane lies at one of a finite number of points known as *critical points*. Leaving aside for the moment the point $z = \infty$, which may or may not be a critical point, it is found that if $f(z)$ is an integral rational function the critical points are given by the roots of the equation $f'(z) = 0$, while if $f(z)$ is a fractional rational function $\varphi(z)/\psi(z)$ (φ and ψ integral and without common factor) the critical points are the roots of the equation $\varphi'\psi - \varphi\psi' = 0$ (accents denoting differentiation). A critical point which occurs p times as a root of one of the above equations is known as a critical point of the pth order. Angles at such a point will correspond to angles in the w-plane $p + 1$ times as great. From this it follows that the point in the w-surface corresponding to a critical point of the pth order will be a branch point at which $p + 1$ leaves of the surface are connected i. e. a branch point of the pth order.

Coming back finally to the point $z = \infty$ we see by making the transformation $z' = 1/z$ that angles at this point either remain unchanged in the

*It seems hardly necessary to say that the number of examples far exceeds the needs of any single person.

w-plane, in which case $z = \infty$ is called an ordinary point, or are multiplied by a positive integer $p + 1$, in which case the point $z = \infty$ is called a critical point of the pth order. If $z = \infty$ is a critical point of the pth order it will of course correspond (like any finite critical point of the pth order) to a branch point of the pth order in the w-surface. Instead of treating the point $z = \infty$ in each case by means of the transformation $z' = 1/z$ it is more convenient to use the general and easily established theorem that the sum of the orders of *all* the critical points of a rational function of the nth degree* is $2n - 2$. This theorem enables us, after having obtained by the method already explained the orders of the finite critical points, to write down at once the order of the point $z = \infty$.

Integral Rational Functions.

Let:

$$w = a_0 z^n + a_1 z^{n-1} + \ldots + a_n \tag{1}$$

(where we suppose that a_0 is not zero) be the rational function for whose inverse the Riemann's surface is to be constructed. The first step is to find the finite critical points as roots of the equation:

$$dw/dz = n a_0 z^{n-1} + (n - 1)\, a_1 z^{n-2} + \ldots + a_{n-1} = 0\,.$$

From the degree of this equation we see that the sum of the orders of the finite critical points is $n - 1$ so that the point $z = \infty$ is a critical point of order $n - 1$. Having computed the values of the critical points we get at once by substituting them in (1) the corresponding branch points. In particular we see that to the critical point $z = \infty$ corresponds the branch point of the $(n - 1)$st order $w = \infty$.

To a given value of w correspond in general n distinct values of z which we obtain by solving (1). If, however, we choose as the value of w a branch point of the pth order $p + 1$ of these values will coincide with the critical point of the z-plane to which the branch point in question corresponds. The remaining $n - p - 1$ values will, however, in general be distinct from each other and from the critical points.† We shall find it convenient to plot in the z-plane not only the critical points but also these non-critical points which correspond to the branch points.

Let us draw lines in the w-plane connecting the finite branch points with

* By the degree of a rational function of z is meant the highest power of z which occurs in either numerator or denominator when the function has been reduced to the form ϕ/ψ.

† It may of course happen (cf. the first example given) that two or more distinct critical points correspond to one and the same value of w i. e. the Riemann's surface may have two or more branch points coincident in position but connecting different sheets. The number of non-critical points corresponding to this value of w will then be correspondingly reduced.

the point $w = \infty$. These lines may be drawn arbitrarily provided that no one crosses itself and no two cross each other. Let us now spread n smooth sheets over the w-plane and cut through all of them along the lines just drawn. The problem is to so reconnect the cut edges as to make the desired Riemann's surface ; the lines along which the sheets were cut thus forming the *junction lines.*

The form above indicated for the junction lines is only one of many possible forms. It may for instance be convenient to connect the finite branch points in succession and connect some point of this line with the point at infinity ; or still other forms may be taken. All that is essential is that 1st the lines do not cut the plane into two distinct parts ; 2nd it should be possible to pass from any branch point to any other branch point by going along the line (after it has been projected onto the sphere). The particular form of junction lines to be chosen depends on the function whose surface is being constructed. For most of the functions which occur in this paper a part of the axis of reals can be used to advantage.

Having drawn the junction lines the next thing to do is to find their *images* in the z-plane, i. e. curves consisting of the aggregate of all those points in the z-plane whose values substituted in (1) will give a value of w lying on a junction line. The method of finding this image can best be described in a concrete example, but it is clear that the image will pass not only through the critical points of the z-plane but also through the other points which correspond to the branch points.

The parts of the image corresponding to the different sections of the junction line should be marked in some distinctive manner, for instance, by different colors, or as in the drawings of this paper by means of dotted lines of various kinds. It will then be found that the z-plane is divided into n distinct parts by means of these dotted (or colored) lines. Each part corresponds to one sheet of the w-surface. The parts are numbered arbitrarily to correspond to the sheets of the w-surface. Having done this the sheets can be reconnected so as to form the desired surface. The manner in which this connection is determined will be explained in detail in the examples given. In getting it the work is facilitated by dividing each sheet of the w-surface into two or more parts by shading, and finding the image of this shading on the z-plane.

Example. Let us form the Riemann's surface for the function

$$w = z^4 - 2\,z^2\,.$$

We must first get the critical points. They are (apart from the critical point $z = \infty$ which we shall not need to consider explicitly) the roots of

$$\frac{dw}{dz} = 4\,(z^3 - z) = 0\,.$$

$\therefore\ z = 0, 1, -1$, are the finite critical points, each of the first order. The critical points, branch points, and values of z corresponding to the branch points are as follows:

Critical point z.	Order.	Branch point w.	Corresponding values of z.
0	1	0	$0, 0, +\sqrt{2}, -\sqrt{2}$
1	1	-1	$+1, +1, -1, -1$.
-1	1	-1	

Plot these values of z in one plane, and the values of w in another plane (see Fig. 1). From the branch points, 0 and -1, draw the junction lines along

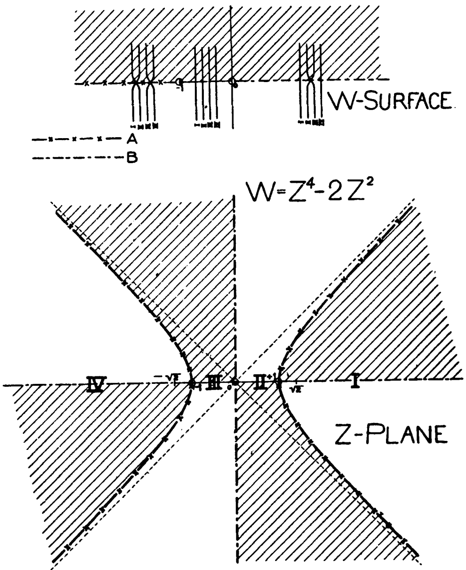

FIGURE 1.

the axis of reals to infinity. In Fig. 1 these have been marked by two kinds of broken lines, A and B. These lines are supposed to be cuts through all four sheets. We must next find the image of these junction lines. To do this find the image of the whole axis of reals.

Let

$$w = u + vi, \; z = x + yi\,.$$

The function becomes

$$\begin{aligned} u + vi &= (x + yi)^4 - 2(x + yi)^2 \\ &= (x^4 - 6x^2y^2 + y^4 - 2x^2 + 2y^2) + (4x^3y - 4xy^3 - 4xy)\,i \\ v &= 4xy\,(x^2 - y^2 - 1)\,. \end{aligned}$$

The equation of the axis of reals of the w-surface is $v = 0$. Therefore the equation of its image is

$$4\,xy\,(x^2 - y^2 - 1) = 0\,.$$

This breaks up into $x = 0$, $y = 0$, and $x^2 - y^2 = 1$. That is, the image on the z-plane of the axis of reals of the w-surface consists of these three loci. This image is next plotted in the z-plane (Fig. 1); it is first put in in ordinary line, and then the marking is determined. Having the image of the whole axis of reals we must find the parts of it which form the image of the two junction lines A and B. This is done by letting z trace out the image curve and following the corresponding movement of w. This movement is readily followed when we remember that angles are preserved except when z is at a critical point, when they are changed in the manner already described. In this example suppose z is very large, real, and positive. Then w is large, real, and positive, and is on the line B. Hence the part of the image in which z is must be marked B. Now let us follow the motion of the two points, giving an arbitrary motion to z, and determining the motion of w.

Let z decrease from its initial value; w will do so also. When $z = \sqrt{2}$, $w = 0$ (see table). As this is an ordinary point of the z-plane angles are preserved, and as z continues to move to the left so must w. w now enters the unmarked region, and therefore z does the same. As z moves to the left from $\sqrt{2}$, w moves to the left from 0. When $z = 1$, $w = -1$. z is now at a critical point of the *first* order, and an angle of the w-plane is double the corresponding angle of the z-plane having its vertex at this point. Hence if the two portions of the z path at $z = 1$ make an angle of 90° the corresponding parts of the w path form an angle of 180°; so that if z turns up or down on the hyperbola at $z = 1$, w will continue to move to the left at $w = -1$, and w enters the region A. As z moves out indefinitely on this branch of the hyperbola, w moves to $-\infty$ along the axis of reals, since there are no more critical points on the hyperbola. Therefore this branch of the hyperbola is marked A. Return now to $z = 1$. If z at this point had made an angle of 180° instead of 90° (i. e. if z had gone straight on to the left), then w would have formed an angle of 360°, and would have turned back at $w = -1$. w

travels then from -1 to 0, and hence from $z = 1$ to $z = 0$ the axis of reals is unmarked. $z = 0$ is a critical point of the first order, and if z at $z = 0$ turns up or down through 90°, w at $w = 0$ will go straight on, or into the region B, since w is now moving to the right; and therefore the whole of the axis of imaginaries of the z-plane must be marked B. The marking of the rest of the image is found in the same manner. Or it may be obtained by revolving that already found through 180° about the origin. This is seen to give the right result by noticing that if for z we substitute $-z$ the value of w is unchanged.

The z-plane is now divided into four distinct regions by the marked lines. In the figure the boundaries of these regions have been made heavier than the rest of the marked lines. The lines B inside of the hyperbola do not bound regions, for it is possible to pass from one side of such a line to the other by going around its end: $\pm \sqrt{2}$. These four regions are numbered arbitrarily I, II, III, IV as shown, and each corresponds to a sheet of the w-surface. Shade the upper half of each of the sheets of the w-surface and get the image of this shading. This is readily done, for the boundary of the shading in the w-surface is the axis of reals and therefore the boundaries of the shading in the z-plane must be the image of the axis of reals. This image divides the z-plane into eight regions, half of which are shaded and the other half unshaded. To determine the shaded regions, let z have a large, real, positive value, and diminish. Then w is large, real, and positive, and is diminishing. w during this motion has the shaded region to the right, and therefore, since angles are unchanged in direction as well as in magnitude, the shaded region is to the right (as z diminishes) of the distant portion of the axis of reals of the z-plane. This shows which part of region I of the z-plane is shaded. If we follow a small closed contour around a branch point of the w-surface the shaded and unshaded regions alternate. Hence in following a small closed contour around a critical point the shaded and unshaded regions must alternate. We can then get all the shading in the z-plane when we have one shaded region by making the shaded and unshaded regions alternate about the critical points. This completes the figure for the z-plane.

Having now the shading for the z-plane, we find how the cuts in the w-surface along A and B must be reconnected. Take the cut B which is along the axis of reals from 0 to $+\infty$. An examination of the z-plane shows that it is possible to cross a line B in going from shaded I to unshaded I; therefore in the w-surface sheet I must be reconnected across the cut B. In the z-plane we can also cross a line B in going from shaded II to unshaded III, and in going from shaded III to unshaded II; hence sheets II and III of the w-surface are connected along the junction line B. In the z-plane we can pass from shaded to unshaded IV across a line B, and sheet IV of the

w-surface is reconnected along the junction line B. This gives the cross-section of the junction line B, as shown in Figure 1. The cross-section of A is found in a similar manner. We have now formed the Riemann's surface for the inverse of the rational function: $w = z^4 - 2z^2$. This Riemann's surface has the peculiarity of having two distinct branch points, one above the other at $w = -1$. The upper branch point with the junction line running out from it connects sheets I and II, while the lower connects sheets III and IV. There is no connection between the upper and the lower pair of sheets along *this* junction line. If, however, we go to junction line B we see that sheets II and III are connected, so that the surface cannot be separated into two parts.

Whenever a Riemann's surface for an integral rational function is to be constructed practically this same work must be performed. In some examples certain steps may offer more difficulty, particularly in the matter of finding the equation of the image of the junction lines, or in plotting the locus of this equation when found. In some cases the function may be such that a very symmetrical surface can be formed, and by noticing this much labor can be saved.

Example. Form the Riemann's surface for

$$w = -z^4 + 4z\,.$$

We have

$$\frac{dw}{dz} = -4\,(z^3 - 1)\,.$$

Then the table giving critical points, etc., is found to be:

Critical point z.	Order.	Branch point w.	Corresponding values of z.
1	1	3	$1, 1, -1 \pm \sqrt{2}i$
ω	1	3ω	$\omega, \omega, (-1 \pm \sqrt{2}i)\,\omega$
ω^2	1	$3\omega^2$	$\omega^2, \omega^2, (-1 \pm \sqrt{2}i)\,\omega^2$

In this table ω is an imaginary cube root of unity. These points are plotted in Fig. 2. Draw the junction lines through the branch points to infinity along lines radiating from the origin. In the figure these have been marked as the lines A, B, C. The image of the whole axis of reals is found to be $y = 0$ and $y^2 = \dfrac{x^3 - 1}{x}$. The curve $y^2 = \dfrac{x^3 - 1}{x}$ consists of three distinct

branches having the axis of imaginaries and two lines through the origin bisecting the angle between the axes as asymptotes. It can easily be picked out in the figure by means of the asymptotes. The marking of the axis of reals and of the right-hand branch of this curve are determined as in the first example. To get started on the other branches suppose that z is very large and pure

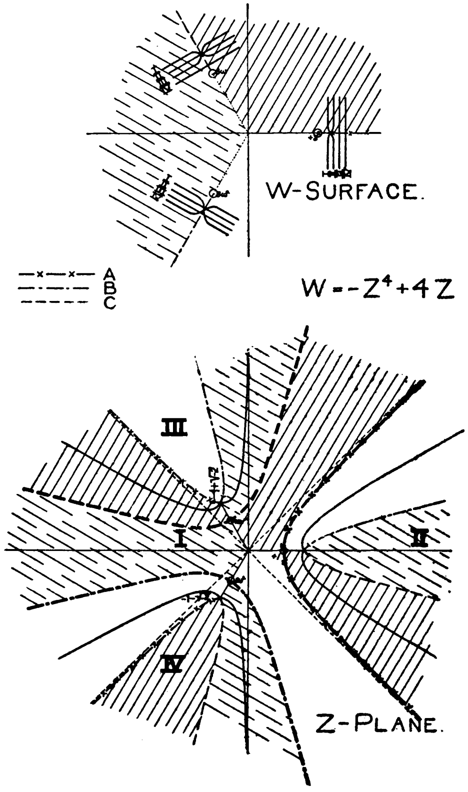

FIGURE 2.

imaginary; then w will be very large real and negative. Now let z move in along the curve (which has the positive half of the axis of imaginaries as asymptote). w will move along the axis of reals from a large negative value toward the right. When $z = -1 + \sqrt{2}i$, $w = 3$. This value of z is not a critical point, so that as z moves on w will go on into the region A, and the rest of the branch must be marked A. The lower branch is marked in the same manner. We have now the whole image of the junction line A. If we proceeded directly to find the image of the other junction lines the work would be difficult. But we see that in the w-plane there is a sort of triangular sym-

metry, that is the w-plane is divided into thirds by the junction lines, and one part can be obtained from the other by rotating about the origin through angles of 120°. This suggests that it may be possible to obtain the image of the other junction lines by revolving the image already found through 120°. That this is the case may be shown as follows. If we let $z' = \omega z$, z' is at the same distance from the origin as z, but the angle of z' is 120° greater than that of z, i. e. z' is obtained from z by rotating through 120° about the origin. Then if

$$w = -z^4 + 4z$$

and

$$w' = -z'^4 + 4z',$$

then

$$w' = -(\omega z)^4 + 4\omega z = \omega(-z^4 + 4z)$$
$$= \omega w.$$

Hence if the z-plane be rotated through an angle of 120° about the origin, the w-plane will be also, and the triangular symmetry of the w-plane extends to the z-plane. Hence to get the image of the junction line C rotate the image of the junction line A through 120° about the origin, and to get the image of junction line B rotate that of A through 240° about the origin. We now have the images of all the junction lines. The z-plane is divided into four and only four distinct parts by the images of the junction lines; that is by the marked parts of the curves. The dotted lines inside of the large branches do not bound regions, as we can pass from one side of such a line to the other by going around its end. These four regions are numbered arbitrarily I, II, III, IV as shown, and each corresponds to a sheet of the w-surface. The boundary lines of the regions are made heavier than the other lines.

The w-surface has been divided into three regions by shading. The image of this shading on the z-plane is readily found by letting z move along the various curves of the z-plane, following the motion of the corresponding point in the w-surface, and putting the shading in the z-plane to the right or left of the path, according as the shading is to the right or left of the w-path. The result is shown in Fig. 2. Having this shading the connections of the sheets along the junction lines can be found at once, and the Riemann's surface is complete.

Problems.

Construct the Riemann's surface for

1. $w = -z^3 + 3z$. [See Klein, *Funk. th.*, p. 72.]
2. $w = -z^5 + 5z$.
3. $w = 2z^5 - 5z^2$.

If there are critical points of order higher than the first the work is prac-

tically the same. Often it is better to use polar coördinates in finding the image of the junction lines. The following is an example of such a function:

$$w = 4z^5 + 5z^4 .$$

$$\frac{dw}{dz} = 20z^3 (z + 1) = 0$$

gives the critical points $z = 0$ of the third order and $z = -1$ of the first order. The branch points and corresponding values of z follow:

Critical point z.	Order.	Branch point w.	Corresponding values of z.
0	3	0	$0, 0, 0, 0, -\frac{5}{4}$
-1	1	$+1$	$-1, -1, +0.61, +0.07 \pm 0.64i$

Since both the branch points lie on the axis of reals we can with advantage take the junction lines along the axis of reals, as in Fig. 3. We therefore must find the image of the axis of reals. If this be done in rectangular coördinates the work is rather long, and even when the equation has been obtained the shape of the curve is not easily seen. Let us use polar coördinates. Then

$$\begin{aligned} z &= r(\cos\varphi + i\sin\varphi) = re^{\phi i} , \\ u + vi &= 4z^5 + 5z^4 = 4r^5e^{5\phi i} + 5r^4e^{4\phi i} \\ &= (4r^5 \cos 5\varphi + 5r^4 \cos 4\varphi) + i(4r^5 \sin 5\varphi + 5r^4 \sin 4\varphi) . \\ v &= r^4(4r \sin 5\varphi + 5 \sin 4\varphi) . \end{aligned}$$

$$4r \sin 5\varphi + 5 \sin 4\varphi = 0 ,$$

is the equation of the image of the axis of reals. It is satisfied by $\varphi = 0$, or the axis of reals of the z-plane, and by the curve

$$r = -\frac{5 \sin 4\varphi}{4 \sin 5\varphi} .$$

This curve evidently has four asymptotes parallel to the lines

$$\varphi = \frac{k\pi}{5} , \; k = 1, 2, 3, 4 .$$

It will be found that these asymptotes all pass through the point $z = -\frac{1}{4}$.* Since the point $z = 0$ is a critical point of the third order the angles formed by the branches of the image passing through this point must be one-fourth as

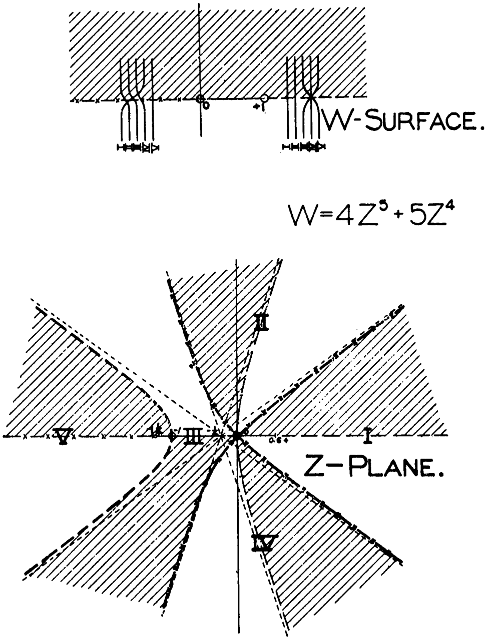

FIGURE 3.

great as the angle in the w-surface. Hence the different branches of the image form angles of 45° with one another at the origin. Having these angles and the asymptotes the curve can be sketched in roughly, and this as a rule is all that is needed in constructing the surface. The carefully drawn curve is shown in Fig. 3. After getting the image the remaining work is the same as in the earlier examples. Of course it must be noticed that when $z = 0$ angles in the w-surface are four times the corresponding angles of the z-plane, and that when $z = -1$ the angles are multiplied by two. In the cross section of the right hand junction line it will be noticed that sheets III and V in running into one another are obliged to pass through sheet IV. We must suppose, however, that IV has no connection whatever with III and V along this junction line. There is no objection to this, but if we wish it can be avoided by numbering the regions of the z-plane differently. Thus if IV be changed into

* Concerning the asymptotes in such cases as this cf. F. Lucas: *Géométrie des polynomes*, Journal de l'Ecole polytechnique 1879.

I, I into II, II into III, and III into IV, sheets IV and V will be connected along the right hand junction line and the other three sheets will lie smoothly above this junction line. The left hand junction line is unchanged by this change of numbering.

Example. Form the Riemann's surface for

$$w = 10z^6 - 24z^5 + 15z^4 .$$

Critical point z.	Order.	Branch point w.	Corresponding values of z.
0	3	0	$0, 0, 0, 0, \dfrac{12 \pm i\sqrt{6}}{10}$
1	2	1	$1, 1, 1, -0.43, -0.08 \pm 0.47i$

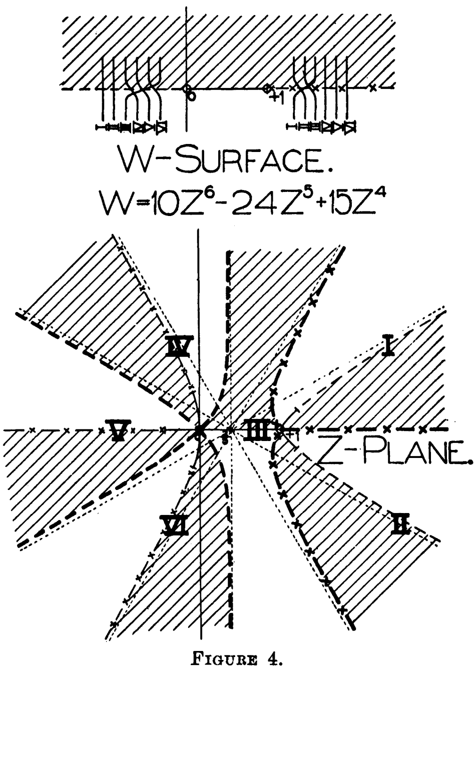

FIGURE 4.

Here again polar coördinates can be used to advantage, and the image of the axis of reals is found to be :

$$10r^2 \sin 6\varphi - 24r \sin 5\varphi + 15 \sin 4\varphi = 0\,.$$

The complete figure is shown in Fig. 4.

The work of constructing a Riemann's surface may often be simplified by the following considerations :

Suppose that the function is of the form

$$w = f(z^k)\,.\,z^l\,, \quad k > l \geqq 0\,, \tag{2}$$

where f is any integral rational function, and k and l are integers. The critical points are given by

$$\frac{dw}{dz} = z^{l-1}\,[lf(z^k) + kz^k f'(z^k)] = z^{l-1}F(z^k) = 0\,. \tag{3}$$

The origin is a critical point of order $(l - 1)$ unless $l = 0$, in which case it is in general of order $(k - 1)$. The remaining critical points are spread on circles having the origin as center, at angular intervals of $\frac{2\pi}{k}$; for if z_0 is a root of (3), az_0 must also be a root, where $a^k = 1$. The branch points corresponding to these critical points will also be spread uniformly on circles. The angular interval will be $\frac{2\pi}{k}$ if l is prime to k. If l is not prime to k the angular interval will be a multiple of $\frac{2\pi}{k}$. If in the w-surface the junction lines from these branch points be so drawn that all can be obtained from one by revolving it about the origin through suitable multiples of $\frac{2\pi}{k}$, their images on the z-plane can be obtained from one another in the same manner, and the figures will have symmetry of the sort that a regular k-sided polygon has, and may therefore be said to have " k-sided polygonal symmetry," or simply " k-symmetry." If l is not prime to k there will be a number of distinct branch points one above the other for certain values of w. If k is even the figure has central symmetry. Use has already been made of this principle in the example $w = -z^4 + 4z = (-z^3 + 4)\,.\,z = f(z^3)\,.\,z$. Here there is triangular symmetry.

Problems. Form the Riemann's surface for

1. $w = 3z^4 + 4z^3$.

2. $w = 3z^5 - 5z^3$.

3. $w = 6z^5 - 15z^4 + 10z^3$.

4. $w = \frac{1}{8}(3z^5 - 10z^3 + 15z)$. [See Klein, *Funk. th.*, p. 78.]

5. $w = z^6 - 3z^4 + 3z^2$.

6. $w = -z^8 + 4z^6 - 6z^4 + 4z^2$.

7. $w = (z^3 - 1)^3 + 1$.

8. $w = \frac{1}{9}(2z^7 - 7z^4 + 14z)$.

9. $w = -\frac{1}{16}(5z^7 - 21z^5 + 35z^3 - 35z)$.

Fractional Rational Functions.

We write these functions in the form $w = \varphi(z)/\psi(z)$ where φ and ψ are integral rational functions without common factor, and determine the finite critical points by means of the equation

$$\varphi'\psi - \varphi\psi' = 0.$$

As the chief difference between this case and the case of integral rational functions already discussed is that the points $z = \infty$ and $w = \infty$ are not necessarily critical and branch points respectively, it will not be necessary to preface the treatment of examples by any general discussion beyond that given in the *Introduction.*

Example. Form the Riemann's surface for

$$w = \frac{3z^4 + 6z^2 - 1}{8z^3}$$

$$\begin{aligned}\varphi'\psi - \varphi\psi' &= 8z^3(12z^3 + 12z) - 24z^2(3z^4 + 6z^2 - 1)\\ &= 24z^2(z^2 - 1)^2.\end{aligned}$$

Hence the finite critical points are 0, 1, -1, each of the second order. The sum of the orders of the finite critical points is $6 = 2(4 - 1)$, and hence $z = \infty$ is not a critical point. The table is :

Critical point z.	Order.	Branch point w.	Corresponding values of z.
0	2	∞	$0, 0, 0, \infty$
1	2	1	$1, 1, 1, -\frac{1}{3}$
-1	2	-1	$-1, -1, -1, +\frac{1}{3}$

Since the branch points are all on the axis of reals the junction lines may be taken along the axis of reals as shown in Fig. 5. The image of the axis of reals is readily found to consist of

$$\varphi = 0 \text{ and } r^2 = 1 \pm \frac{\sin\varphi}{\sin 60^\circ}.$$

The different branches of this image form angles of 60° at 0, 1, — 1.

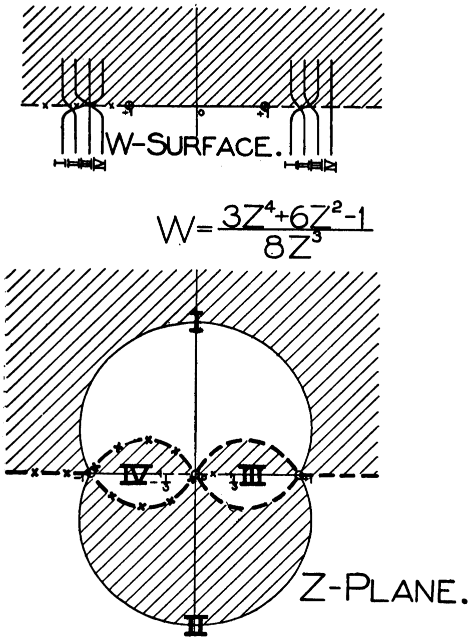

FIGURE 5.

The work of forming the surface is precisely the same as in the earlier examples, and should offer no difficulty. The result is given in Fig. 5.

To show the effect of drawing the junction lines for one and the same function in various ways consider:

$$w = \frac{z^4 + 2z}{2z^3 + 1} = \frac{z^3 + 2}{2z^3 + 1} \cdot z \, .$$

The critical points, etc., are as follows, ω being a complex cube root of unity:

Critical point z.	Order.	Branch point w.	Corresponding values of z.
1	2	1	$1, 1, 1, -1$
ω	2	ω	$\omega, \omega, \omega, -\omega$
ω^2	2	ω^2	$\omega^2, \omega^2, \omega^2, -\omega^2$

The form of the function shows that there is triangular symmetry. In

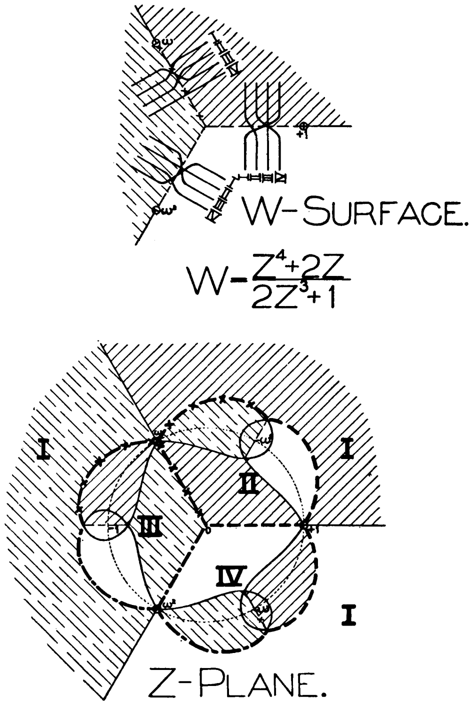

FIGURE 6.

the w-surface draw lines from the origin to each of the branch points as the first method of drawing the junction lines (Fig. 6). Since part of the junction line lies along the axis of reals, we must find the image of the axis of reals. It is found to consist of

$$\varphi = 0 \text{ and } r^3 = 3 \cos \varphi - 2 \cos^3 \varphi \pm \sin \varphi \sqrt{4 \cos^2 \varphi - \cos^2 2\varphi}.$$

The curve is that one of the three shown in Fig. 6 which passes through $+1$. The other two are obtained by revolving this one through angles of 120° and 240° respectively. The figure is completed in the usual manner. It should be carefully noted that the origin of the w-surface is *not* a branch point. This method of drawing the junction lines from some point not a branch point to the various branch points frequently adds to the symmetry of the figure.

For this same function $w = \dfrac{z^4 + 2z}{2z^3 + 1}$ the junction lines might have been

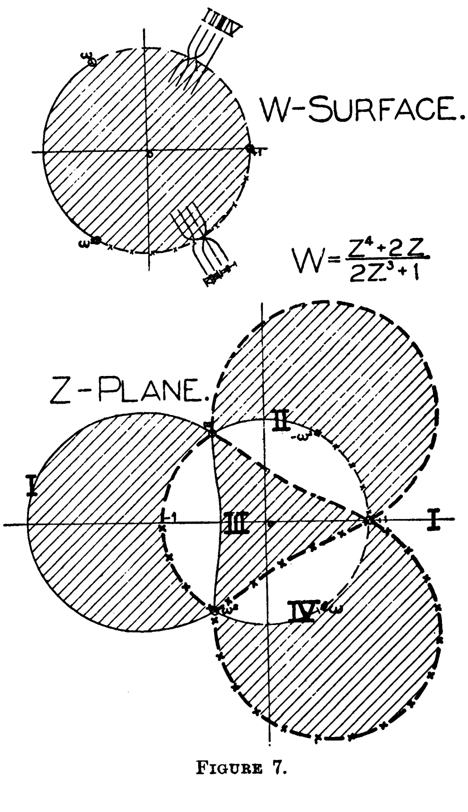

FIGURE 7.

drawn along the unit circle, as indicated in Fig. 7. To get the image of the unit circle proceed as follows:

$$w = u + vi\,, \quad z = re^{\phi i}\,,$$

$$u + vi = \frac{r^4 e^{4\phi i} + 2re^{\phi i}}{2r^3 e^{3\phi i} + 1}\,,$$

$$\therefore \qquad u - vi = \frac{r^4 e^{-4\phi i} + 2re^{-\phi i}}{2r^3 e^{-3\phi i} + 1}\,,$$

$$(u + vi)(u - vi) = u^2 + v^2 = \frac{r^8 + 2r^5(e^{3\phi i} + e^{-3\phi i}) + 4r^2}{4r^6 + 2r^3(e^{3\phi i} + e^{-3\phi i}) + 1}\,,$$

$$\therefore \qquad u^2 + v^2 = \frac{r^8 + 4r^5 \cos 3\varphi + 4r^2}{4r^6 + 4r^3 \cos 3\varphi + 1}\,.$$

The equation of the unit circle in the w-surface is $u^2 + v^2 = 1$. Hence the equation of its image is

$$\frac{r^8 + 4r^5 \cos 3\varphi + 4r^2}{4r^6 + 4r^3 \cos 3\varphi + 1} = 1\,,$$

or

$$(r^8 - 1) + 4r^3(r^2 - 1)\cos 3\varphi - 4r^2(r^4 - 1) = 0\,.$$

Since $(r^2 - 1)$ is a factor of this it follows that the unit circle of the z-plane forms part of the image of the unit circle of the w-surface. The remainder of the image is the curve

$$(r^2 + 1)(r^4 - 4r^2 + 1) + 4r^3 \cos 3\varphi = 0\,.$$

This curve is plotted in Fig. 7. It has three loops and passes through the points 1, ω, ω^2, at which points it forms angles of 60° with the unit circle. The next thing is to get the parts of this image corresponding to the two junction lines of the w-surface. After getting started on the marking the rest of it is found in the same manner as in the earlier examples. To get started let z move from $z = +1$ in a positive direction around the unit circle. w will then start from $w = 1$ and move in one direction or the other about its unit circle. To determine which direction we observe that when z passes through $1, -\omega^2$, and ω in succession w passes through $1, +\omega^2$ and ω in succession. Therefore when z starts out in a positive direction from $z = +1$, w starts out in a negative direction from $w = +1$. Knowing this the rest of the marking and shading is put in by using the principles already given. A comparison of Fig.

6 and Fig. 7 will show the wide difference which can be made in the appearance of the Riemann's surface, and in the corresponding division of the z-plane by the choice of junction lines.

As a final example take

$$w = \frac{z^3 + \sqrt{3}z}{\sqrt{3}z^2 + 1}.$$

The finite critical points are the roots of

$$z^4 + 1 = 0,$$

or

$$e^{\frac{\pi i}{4}},\ e^{\frac{3\pi i}{4}},\ e^{\frac{5\pi i}{4}},\ \text{and}\ e^{\frac{7\pi i}{4}},$$

each of these being of the first order. The sum of the orders is $4 = 2\,(3 - 1)$, and hence $z = \infty$ is not a critical point. After rather long reductions we get the following table :

Critical point z.	Order.	Branch point w.	Corresponding values of z.
$e^{\frac{\pi i}{4}}$	1	$e^{\frac{\pi i}{12}}$	$e^{\frac{\pi i}{4}},\ e^{\frac{\pi i}{4}},\ e^{\frac{19\pi i}{12}}$
$e^{\frac{3\pi i}{4}}$	1	$e^{\frac{11\pi i}{12}}$	$e^{\frac{3\pi i}{4}},\ e^{\frac{3\pi i}{4}},\ e^{\frac{17\pi i}{12}}$
$e^{\frac{5\pi i}{4}}$	1	$e^{\frac{13\pi i}{12}}$	$e^{\frac{5\pi i}{4}},\ e^{\frac{5\pi i}{4}},\ e^{\frac{7\pi i}{12}}$
$e^{\frac{7\pi i}{4}}$	1	$e^{\frac{23\pi i}{12}}$	$e^{\frac{7\pi i}{4}},\ e^{\frac{7\pi i}{4}},\ e^{\frac{5\pi i}{12}}$

These points are easily plotted, for $e^{\phi i}$ is that point of the unit circle whose angle is φ. Since all the branch points are on the unit circle, draw the junction lines along the unit circle (Fig. 8). The image of the unit circle is found to be the unit circle of the z-plane and the curve

$$r^2 = 1 - \sqrt{3}\cos 2\varphi \pm \sqrt{3\cos^2 2\varphi - 2\sqrt{3}\cos 2\varphi}.$$

The figure is then easily completed.

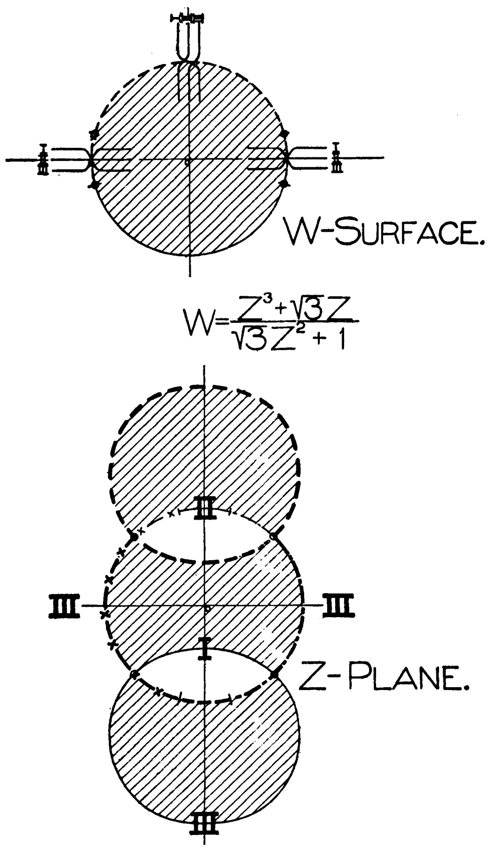

FIGURE 8.

Problems. Construct the Riemann's surface for :

1. $w = \dfrac{2z^3 + 1}{3z^2}$.

*2. $w = \dfrac{3z^4 + 1}{4z^3}$.

3. $w = \dfrac{4z^5 + 1}{5z^4}$.

4. $w = \dfrac{2z^5 + 3}{5z^4}$.

5. $w = \dfrac{-z^3 + 3z^2}{3z - 1}$.

6. $w = \dfrac{z^3 + 3z}{3z^2 + 1}$.

7. $w = \dfrac{-z^4 + 2z^3}{2z - 1}$.

8. $w = \dfrac{-z^5 + 5z^3}{5z^2 - 1}$.

* Cf. Holzmüller's *Isogonale Verwandtschaften*, p. 224.

9. $w = \dfrac{-3z^5 + 5z^4}{5z - 3}$.

10. $w = \dfrac{z^5 + 5z^2}{5z^3 + 1}$.

11. $w = \dfrac{z^3 + i\sqrt{3}z}{\sqrt{3}z^2 + i}$.

12. $w = \dfrac{z^3 + i\sqrt{3}z}{\sqrt{3}z^2 - i}$.

13. $w = \dfrac{8z}{z^4 - 6z^2 - 3}$.

14. $w = \dfrac{z^3 + 2}{3z}$.

15. $w = \dfrac{z^4 + 3}{4z}$.

16. $w = \dfrac{3z^5 + 5z}{5z^4 + 3}$.

17. $w = \dfrac{5z^6 + 5z^3 - 1}{9z^5}$.

18. $w = \dfrac{2z^6 + 8z^3 - 1}{9z^4}$.

19. $w = \dfrac{(3z - 1)(z + 1)^3}{(3z + 1)(z - 1)^3}$.

20. $w = \dfrac{z^5 + \sqrt{5}z^3}{\sqrt{5}z^2 - 1}$.

21. $w = \dfrac{2z^5 - (1 + i)\sqrt{6}z^4 + 4iz^3}{4z^2 + (1 + i)\sqrt{6}z + 2i}$.

22. $w = \dfrac{z^4 + 1}{2z^2}$.

23. $w = \dfrac{z^4 - (1 + i)z^3 + 3iz^2}{3z^2 + (1 + i)z + i}$.

An important class of functions is that known as the "Double Pyramid Functions." The general form is $w = \frac{1}{2}(z^n + z^{-n})$. These functions are so called because if the junction lines be taken along the axis of reals and the resulting z-plane be projected on the sphere, this sphere will be divided as by a regular inscribed double pyramid. (See Klein, *Funk. th.*, pp. 97–103.)

Problem. Form the Riemann's surface for

$$w = \frac{z^{10} + 1}{2z^5}.$$

These functions form only the simplest case of the *functions of the regular solids* concerning which the reader may be referred to Forsyth's *Theory of Functions*, pp. 564–573, and to Klein's *Vorlesungen über das Ikosaeder.*

We close with a few remarks of a general nature:

Two functions may be said to be of the same *type* when one can be transformed into the other by linear transformations of w and z. A linear transformation is one of the form $z_1 = \dfrac{\alpha z + \beta}{\gamma z + \delta}$.* A transformation of this kind

* For a full discussion of the linear function see a paper by Cole, Annals of Mathematics, Vol. V.

does not change the angle between any two intersecting curves, and a circle (of finite or infinite radius) is changed into a circle. Although in the plane a figure may be completely changed by a transformation of this kind, on the sphere its appearance will not be *materially* changed, even though its size and shape in general are changed. Linear transformations of the dependent and independent variables simply move the critical and branch points without changing their number or order, do not change the connection of sheets along junction lines, and two distinct points cannot be made to coincide. If then we know the Riemann's surface for any function, the general form of the curves on the two spheres for all functions of the same type is known, and by projecting the spheres back on the plane the general form of the Riemann's surface for all functions of the same type is at once obtained. Those transformations which simply rotate the sphere about some diameter are particularly useful. The transformation $z_1 = \frac{1}{z}$ rotates the sphere 180° about the diameter joining $+1, -1$. If we compare examples 1 and 14 of the preceding list, we see that one is obtained from the other by substituting $\frac{1}{z}$ for z; therefore if we have the figure for either function the figure for the other can be obtained by rotating the z-sphere through 180 about the diameter joining $+1$ and -1, and projecting the figure back on the plane. The w-surface is to be left unchanged. It may be necessary to change both w and z; for example if we take $w_1 = z_1^3$ and substitute

$$w_1 = \frac{w-1}{w+1}, \quad z_1 = \frac{z-1}{z+1};$$

the result is

$$w = \frac{z^3 + 3z}{3z^2 + 1}. \qquad \text{[See example 6.]}$$

The transformation $w_1 = \dfrac{w-1}{w+1}$ rotates the w_1 sphere through 90° about the axis joining $+i, -i$, so that 0 goes into -1, etc. If for $w_1 = z_1^3$ the junction line be taken from 0 to ∞ along the positive half of the axis of reals, for the transformed function the junction line will still be along the axis of reals, but will join -1 and $+1$. For the z_1 sphere the image of the axis of reals consists of three meridians which divide the sphere into six equal lunes, alternately shaded and unshaded. Revolving this sphere and projecting on the z-plane we get as the dividing lines of that plane the axis of reals, and two circles, passing through $+1, -1$, and forming angles of 60° with the axis of reals at these points. The marking of the lines and the shading are also obtained from the same projection. Thus when once we know that two func-

tions are linearly transformable into one another we can very easily get the Riemann's surface and the corresponding division of the z-plane of one from that of the other.

General Formulæ.

All of the examples given in the preceding pages are special cases of some of the following functions, or linear transformations of the special cases. It would be difficult to form the Riemann's surface for most of the functions except in the simpler cases, but the critical points and branch points are always easily found, and often after some of the simple cases have been constructed the general form of the more complicated surfaces given by the same formula is evident.

In all of the functions k, m, n, p, are any positive integers subject to the conditions given.

I. *All* of the *integral* rational functions given are included in

$$w = \frac{(-1)^p\, n\, (n-m)\, (n-2m) \ldots (n-pm)}{p!\, m^p} \sum_{k=0}^{k=p} \frac{(-1)^k p\, (p-1) \ldots (p-k+1)}{k!\, (n-km)} z^{n-km}$$

where $pm \leqq (n-1)$. The finite critical points are the roots of

$$z^{n-pm-1}\, (z^m - 1)^p = 0\, .$$

These critical points are uniformly spaced on the unit circle, and the origin is a critical point. The branch points are uniformly spaced on the unit circle, and at the origin.

The writer has not been able to bring all the fractional functions into a single form, but the following list includes all the examples given, and some for which no special cases have been given.

II.

$$w = \frac{(-1)^p m(m-n)(m-2n) \ldots (m-pn) \sum_{k=0}^{k=p} \frac{(-1)^k p(p-1) \ldots (p-k+1)}{k!\, (m-kn)} z^{(p-k)n}}{p!\, n^p\, z^{pn-m}}$$

where $1 \leqq m \leqq pn - 1$, and $m \neq kn$ for $k = 1, 2, \ldots (p-1)$. The critical points are given by

$$z^{pn-m-1}\, (z^n - 1)^p = 0\, ,$$

and $z = \infty$ is a critical point of order $m - 1$. The function is of degree np. The branch points are on the unit circle, and at $w = \infty$. If $p = 1$ and $n = 2m$ this reduces to the double pyramid function.

III.

$$w = \frac{\sum_{k=0}^{k=m-1} \frac{1}{(n-k)(n-k-1)} \left[\frac{n-m}{n}\right]^{\frac{k}{m}} z^{n-k}}{z - \left[\frac{n-m}{n}\right]^{\frac{1}{m}}}$$

where $1 \leqq m \leqq n$. The critical points are given by

$$z^{n-m}(z^m - 1) = 0\,,$$

and $z = \infty$ is a critical point of order $(n-2)$. The same determination of $\left[\frac{n-m}{m}\right]^{\frac{1}{m}}$ is to be used throughout.

IV.

$$w = \frac{\sqrt{n-2m}\,.\,z^n + \sqrt{n}\,z^{n-m}}{\sqrt{n}\,.\,z^m - \sqrt{n-2m}}$$

where $1 \leqq m \leqq (n-1)$, and $2m \ddagger n$. The critical points are given by

$$z^{n-m-1}(z^{2m} - 1) = 0\,,$$

and $z = \infty$ is a critical point of order $(n-m-1)$.

V.

$$w = \frac{(n-3)z^n - (1+i)\sqrt{n^2-4n+1}\,.\,z^{n-1} + i(n-1)z^{n-2}}{(n-1)z^2 + (1+i)\sqrt{n^2-4n+1}\,.\,z + i(n-3)}$$

where $n > 3$. The critical points are given by

$$z^{n-3}(z^4 - 1) = 0\,,$$

and $z = \infty$ is a critical point of order $(n-3)$.

VI.

$$w = \frac{(2m-n)\,z^n + nz^{n-m}}{nz^m + (2m-n)}$$

where $1 \leqq m \leqq (n-1)$, and $2m \ddagger n$. The critical points are given by

$$z^{n-m-1}(z^m - 1)^2 = 0\,,$$

and $z = \infty$ is a critical point of order $(n-m-1)$. The branch points are on the unit circle, and at 0 and ∞.

VII.

$$w = \frac{(n-3m)(n-4m)z^n - 2n(n-4m)z^{n-m} + n(n-m)z^{n-2m}}{n(n-m)z^{2m} - 2n(n-4m)z^m + (n-3m)(n-4m)}$$

where $1 \leqq m \leqq \frac{n-1}{2}$; $m \neq \frac{n}{k}$ for $k = 3, 4$; $n \geqq 5$. The critical points are given by

$$z^{n-2m-1}(z^m - 1)^4 = 0,$$

and $z = \infty$ is a critical point of order $(n - 2m - 1)$. The branch points are on the unit circle, at 0 and at ∞.

VIII. Both VI and VII are included in the following form, but the writer has not yet determined the critical points.

$$w = \frac{\sum\limits_{k=0}^{k=p} \frac{(-1)^k z^{n-km}}{(p-k)!\,k!\,[n-km]\,[n-(k+1)m]\,\ldots\,[n-(k+p)m]}}{\sum\limits_{k=0}^{k=p} \frac{(-1)^k z^{(p-k)m}}{(p-k)!\,k!\,[n-(p-k)m]\,[n-(p-k+1)m]\ldots[n-(2p-k)m]}}$$

where $1 \leqq m \leqq \frac{n-1}{p}$; $m \neq \frac{n}{p+k}$ for $k = 1, 2, \ldots p$; and $n \geqq 2p + 1$.

It seems probable that the critical points are given by

$$z^{n-pm-1}(z^m - 1)^{2p} = 0.$$

$z = \infty$ is a critical point of order $(n - pm - 1)$.

Harvard University, 1896.